

THE GENERALISED NILRADICAL OF A LIE ALGEBRA

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Abstract

A solvable Lie algebra L has the property that its nilradical N contains its own centraliser. This is interesting because gives a representation of L as a subalgebra of the derivation algebra of its nilradical with kernel equal to the centre of N . Here we consider several possible generalisations of the nilradical for which this property holds in any Lie algebra. Our main result states that for every Lie algebra L , $L/Z(N)$, where $Z(N)$ is the centre of the nilradical of L , is isomorphic to $\text{Der}(N^*)$ where N^* is an ideal of L such that N^*/N is the socle of a semisimple Lie algebra.

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1 Introduction

Throughout, L will be a finite-dimensional Lie algebra, over a field F , with nilradical N and radical R . If L is solvable, then N has the property that $C_L(N) \subseteq N$. This property supplies a representation of L as a subalgebra of $\text{Der}(N)$ with kernel $Z(N)$. The purpose of this paper is to seek a larger ideal for which this property holds in all Lie algebras. The corresponding problem has been considered for groups (see, for example, Aschbacher [1, Chapter 11]). In group theory, the quasi-nilpotent radical (also called by some the generalised Fitting subgroup), $F^*(G)$, of a group G is defined to be $F(G) + E(G)$, where $F(G)$ is the Fitting subgroup and $E(G)$ is the set of *components* of G : that is, the quasi-simple subnormal subgroups of the group. It is also equal to the socle of $C_G(F(G))F(G)/F(G)$. The generalised Fitting subgroup, $\tilde{F}(G)$, is defined to be the socle of $G/\Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of G (see, for example, [8]). Here we consider various possible analogues for Lie algebras.

First we introduce some notation that will be used. The *centre* of L is $Z(L) = \{x \in L : [x, y] = 0 \text{ for all } y \in L\}$; if S is a subalgebra of L , the *centraliser* of S in L is $C_L(S) = \{x \in L : [x, S] = 0\}$; the *Frattini ideal*, $\phi(L)$, of L is the largest ideal contained in all of the maximal subalgebras of L ; we say that L is ϕ -free if $\phi(L) = 0$; the *socle* of S , $\text{Soc } S$, is the sum of all of the minimal ideals of S ; and the *L -socle* of S , $\text{Soc}_L S$, is the sum of all of the minimal ideals of L contained in S . The symbol ' \oplus ' will be used to denote an algebra direct sum, whereas ' $\dot{+}$ ' will denote a direct sum of the vector space structure alone.

We call L *quasi-simple* if $L^2 = L$ and $L/Z(L)$ is simple. Of course, over a field of characteristic zero a quasi-simple Lie algebra is simple, but that is not the case over fields of prime characteristic. For example, A_n where $n \equiv -1 \pmod{p}$ is quasi-simple, but not simple. This suggests using the quasi-simple subideals of a Lie algebra L to define a corresponding $E(L)$. However, first note that quasi-simple subideals of L are ideals of L . This follows from the following easy lemma.

Lemma 1.1 *If I is a perfect subideal (that is, $I^2 = I$) of L then I is a characteristic ideal of L .*

Proof. If I is perfect then $I = I^n$ for all $n \in \mathbb{N}$. It follows that $[L, I] = [L, I^n] \subseteq L$ (ad I) $^n \subseteq I$ for some $n \in \mathbb{N}$, and hence that I is an ideal of L . But now, if $D \in \text{Der}(L)$, then $D([x_1, x_2]) = [x_1, D(x_2)] + [D(x_1), x_2] \in I$ for all $x_1, x_2 \in I$. Hence $D(I) = D(I^2) \subseteq I$. \square

Combining this with the preceding remark we have the following.

Lemma 1.2 *Let L be a Lie algebra over a field of characteristic zero. Then I is a quasi-simple subideal of L if and only if it is a simple ideal of L .*

We say that an ideal A of L is *quasi-minimal* in L if $A/Z(A)$ is a minimal ideal of $L/Z(A)$ and $A^2 = A$. Clearly a quasi-simple ideal is quasi-minimal. Over a field of characteristic zero, an ideal A of L is quasi-minimal if and only if it is simple. So an alternative is to define $E(L)$ to consist of the quasi-minimal ideals of L . We investigate these two possibilities in sections 3 and 5.

In sections 4 and 6 our attention turns to two further candidates for a generalised nilradical: the L -socle of $(N + C_L(N))/N$ and the socle of $L/\phi(L)$. All of these possibilities turn out to be related, but not always equal.

2 Preliminary results

Let L be a Lie algebra over a field F and let U be a subalgebra of L . If F has characteristic $p > 0$ we call U *nilregular* if the nilradical of U , $N(U)$, has nilpotency class less than $p - 1$. If F has characteristic zero we regard every subalgebra of L as being nilregular. We say that U is *characteristic in L* if it is invariant under all derivations of L . Then we have the following result.

Theorem 2.1 (i) *If I is a nilregular ideal of L then $N(I) \subseteq N(L)$.*

(ii) *If I is a nilregular subideal of L and every subideal of L containing I is nilregular, then $N(I) \subseteq N(L)$.*

Proof.

- (i) We have that $N(I)$ is characteristic in I . This is well-known in characteristic zero, and is given by [7, Corollary 1] in characteristic p . Hence it is a nilpotent ideal of L and the result follows.
- (ii) Let $I = I_0 < I_1 < \dots < I_n = L$ be a chain of subalgebras of L with I_j an ideal of I_{j+1} for $j = 0, \dots, n - 1$. Then $N(I) \subseteq N(I_1) \subseteq \dots \subseteq N(I_n) = N(L)$, by (i).

□

Similarly, we will call the subalgebra U *solregular* if the underlying field F has characteristic zero, or if it has characteristic p and the (solvable) radical of U , $R(U)$, has derived length less than $\log_2 p$. Then we have the following corresponding result.

Theorem 2.2 (i) *If I is a solregular ideal of L then $R(I) \subseteq R(L)$.*

(ii) *If I is a solregular subideal of L and every subideal of L containing I is solregular, then $R(I) \subseteq R(L)$.*

Proof. This is similar to the proof of Theorem 2.1, using [9, Theorem 2].

□

We also have the following result which we will improve upon below, but by using a deeper result than is required here.

Theorem 2.3 *Let L be a Lie algebra over a field F , and let I be a minimal non-abelian ideal of L . Then either*

(i) *I is simple or*

(ii) *F has characteristic p , $N(I)$ has nilpotency class greater than or equal to $p - 1$, and $R(I)$ has derived length greater than or equal to $\log_2 p$.*

Proof. Let I be a non-abelian minimal ideal of L and let J be a minimal ideal of I . Then $J^2 = J$ or $J^2 = 0$. The former implies that J is an ideal of L by Lemma 1.1, and hence that I is simple. So suppose that $J^2 = 0$. Then $N(I) \neq 0$ and $R(I) \neq 0$. But if I is nilregular we have that $N(I) \subseteq N(L) \cap I = 0$, since I is non-abelian, a contradiction. Similarly, if I is solregular, then $R(I) \subseteq R(L) \cap I = 0$, a contradiction. The result follows. □

As a result of the above we will call the subalgebra U *regular* if it is either nilregular or solregular; otherwise we say that it is *irregular*. Then we have the following corollary.

Corollary 2.4 *Let L be a Lie algebra over a field F . Then every minimal ideal of L is abelian, simple or irregular.*

Block's Theorem on differentiably simple rings (see [3]) describes the irregular minimal ideals as follows.

Theorem 2.5 *Let L be a Lie algebra over a field of characteristic $p > 0$ and let I be an irregular minimal ideal of L . Then $I \cong S \otimes \mathcal{O}_n$, where S is simple and \mathcal{O}_n is the truncated polynomial algebra in n indeterminates. Moreover, $N(I)$ has nilpotency class $p - 1$ and $R(I)$ has derived length $\lceil \log_2 p \rceil$.*

Proof. Every non-abelian minimal ideal I of L is $\text{ad}|_I (L)$ -simple, so the first assertion follows from [3, Theorem 1]. Now $N(I) = R(I) \cong S \otimes \mathcal{O}_n^+$, where \mathcal{O}_n^+ is the augmentation ideal of \mathcal{O}_n . It is then straightforward to check that the final assertion holds. \square

Note that if \mathcal{N} and \mathcal{S} are the classes of Lie algebras that are themselves nilregular and solregular respectively, then $\mathcal{N} \not\subseteq \mathcal{S}$ and $\mathcal{S} \not\subseteq \mathcal{N}$, as the following examples show.

EXAMPLE 2.1 *Let L be a filiform nilpotent Lie algebra of dimension n over a field F . Then L has nilpotency class $n - 1$ and derived length 2. Thus, if F has characteristic $p > 3$, and $n \geq p$, then L has nilpotency class greater than or equal to $p - 1$, and so is not nilregular. However, it is solregular, since $2 < \log_2 p$.*

EXAMPLE 2.2 *Let $L = Fe_1 + Fe_2$ with product $[e_1, e_2] = e_2$ and let F have characteristic 3. The $N(L) = Fe_2$ has nilpotency class $1 < p - 1$ and so L is nilregular. But $R(L) = L$, so L has derived length $2 > \log_2 p$ and is not solregular.*

For every Lie algebra $L^{(n)} \subseteq L^{2^n}$, so any nilregular nilpotent Lie algebra of nilpotency class 2^n is solregular, since $2^n < p - 1 < p$ implies that $n < \log_2 p$. However, it is not true generally that a nilpotent nilregular Lie algebra is solregular, as the following example shows.

EXAMPLE 2.3 *Let L be the seven-dimensional Lie algebra over a field F of characteristic $p = 7$ with basis e_1, \dots, e_7 and products $[e_2, e_1] = e_4$, $[e_3, e_1] = e_5$, $[e_3, e_2] = e_5$, $[e_4, e_3] = -e_6$, $[e_5, e_1] = e_7$, $[e_5, e_2] = 2e_6$, $[e_5, e_4] = e_7$, $[e_6, e_1] = e_7$ and $[e_6, e_2] = e_7$ (see [4, page 87]). Then L has nilpotency class $5 < p - 1$ and so is nilregular, but its derived length is $3 > \log_2 p$, so it is not solregular.*

We also have the following result.

Corollary 2.6 *If L is a Lie algebra and A is a regular ideal of L , then A is quasi-minimal in L if and only if it is a quasi-simple ideal of L .*

However, the above result is not true for all ideals, as the following example shows.

EXAMPLE 2.4 Let $L = sl(2) \otimes \mathcal{O}_m + 1 \otimes \mathcal{D}$, where \mathcal{O}_m is the truncated polynomial algebra in m indeterminates, \mathcal{D} is a non-zero solvable subalgebra of $Der(\mathcal{O}_m)$, \mathcal{O}_m has no \mathcal{D} -invariant ideals, and the ground field is algebraically closed of characteristic $p > 5$. Then L is semisimple and $A = sl(2) \otimes \mathcal{O}_m$ is the unique minimal ideal of L (see [18, Theorem 6.4]). Since $Z(A) = 0$, A is clearly quasi-minimal but not quasi-simple.

If S is a subalgebra of L , we denote by $R_c(S)$ the (solvable) characteristic radical of S ; that is, the sum of all of the solvable characteristic ideals of L . (see Seligman [11]).

Theorem 2.7 Let L be a Lie algebra over any field F . Then $R_c(C_L(N)) = Z(N)$. Moreover, if $C_L(N)$ is regular, then $R_c(C_L(N)) = R(C_L(N))$.

Proof. Let $Z = Z(N)$, $\overline{L} = L/Z$ and $H = R_c(C_L(N))$. Then H is a characteristic ideal of $C_L(N)$, and hence an ideal of L . Assume that $\overline{H} \neq 0$. Then there exists $k \geq 1$ such that $H^{(k+1)} \subseteq Z$ but $X = H^{(k)} \not\subseteq Z$. Then $X^2 \subseteq Z$ and $X^3 \subseteq [N, C_L(N)] = 0$, since $X \subseteq C_L(N)$. It follows that X is a nilpotent ideal of L , and hence that $X \subseteq N$. But $[X, N] = 0$, giving $X \subseteq Z$, a contradiction.

Now suppose that $C_L(N)$ is nilregular. Then, clearly, $R_c(C_L(N)) \subseteq R(C_L(N))$. Suppose that $R(C_L(N)) \neq Z$. Let A/Z be a minimal ideal of $C_L(N)/Z$ with $A \subseteq R(C_L(N))$. Then $A^3 = 0$ and so $A \subseteq N(C_L(N)) \subseteq N(L)$, by Theorem 2.1 (i). Hence $A = Z$, a contradiction.

Finally, suppose that $C_L(N)$ is solregular. Then $R(C_L(N)) = R(L) \cap C_L(N)$ is an ideal of L , and arguing as in the first paragraph of this proof shows that $R(C_L(N)) = Z(N)$. \square

This has the following useful corollary.

Corollary 2.8 Let L be a Lie algebra over a field F , let N be its nilradical and let $C = C_L(N)$ be regular. Then

- (i) if $\phi(C) \cap Z(N) = 0$, $C = Z(N) \dot{+} B$ where B is a semisimple subalgebra of L and B^2 is an ideal of L ;
- (ii) if $\phi(L) \cap Z(N) = 0$, $C = Z(N) \oplus B$ where B is a maximal semisimple ideal of L ; and

(iii) if F has characteristic zero, then $C = Z(N) \oplus S$ where S is the maximal semisimple ideal of L .

Proof.

- (i) Suppose that $\phi(C) \cap Z(N) = 0$. Then $C = Z(N) \dot{+} B$ for some subalgebra B of C , by [14, Lemma 7.2]. Moreover, $B \cong C/Z(N)$ is semisimple, by Theorem 2.7, and $B^2 = C^2$ is an ideal of L .
- (ii) Suppose that $\phi(L) \cap Z(N) = 0$. The $L = Z(N) \dot{+} U$ for some subalgebra U of L , by [14, Lemma 7.2] again. It follows that $C = Z(N) \oplus B$ where $B = C \cap U$, which is an ideal of L , and B is semisimple. Moreover, if S is a semisimple ideal of L with $B \subseteq S$, then $[S, N] \subseteq S \cap N = 0$, so $S \subseteq C$. Hence $S = B$.
- (iii) So suppose now that F has characteristic zero. Then $C = Z(N) \dot{+} B$ where B is a Levi factor of C . Also, $B = B^2 = C^2$ is an ideal of L , so $C = Z(N) \oplus B$. Moreover, if S is the maximal semisimple ideal of L , then $B \subseteq S$ and $[S, N] \subseteq S \cap N = 0$, so $S \subseteq C$. It follows that $S = B$.

□

Finally, the following straightforward results will prove useful.

Lemma 2.9 *Let K be an ideal of L with $K \subseteq C_L(N)$. Then $Z(K) = Z(N) \cap K$.*

Proof. Clearly $Z(K)$ is an abelian ideal of L , so $Z(K) \subseteq N$. Moreover, $[Z(K), N] \subseteq [K, N] = 0$, so $Z(K) \subseteq Z(N) \cap K$. Also $[Z(N) \cap K, K] \subseteq [N, K] = 0$, so $Z(N) \cap K \subseteq Z(K)$. □

Lemma 2.10 *Let L be any Lie algebra and suppose that A is an ideal of L with $A^2 = A$. Then $Z(A) \subseteq \phi(L)$. If A is a quasi-minimal ideal of L , then $Z(A) = A \cap \phi(L)$.*

Proof. Suppose that $Z(A) \not\subseteq \phi(L)$. Then there is a maximal subalgebra U of L such that $L = Z(A) + U$. Thus $A = Z(A) + U \cap A$ and $U \cap A$ is an ideal of L . It follows that $A = A^2 = (U \cap A)^2 \subseteq U \cap A \subseteq A$, whence $Z(A) \subseteq U$, a contradiction. Hence $Z(A) \subseteq \phi(L)$.

Suppose now that A is a quasi-minimal ideal of L . Then $Z(A) \subseteq A \cap \phi(L) \subseteq A$, so $A \cap \phi(L) = A$ or $Z(A)$. The former implies that $A \subseteq \phi(L)$, which is impossible since $\phi(L)$ is nilpotent. Hence $A \cap \phi(L) = Z(A)$. □

3 The quasi-minimal radical

Here we construct a radical by adjoining the quasi-minimal ideals of L to its nilradical N .

Lemma 3.1 *Quasi-minimal ideals of L are characteristic in L .*

Proof. This follows from Lemma 1.1. \square

Lemma 3.2 *Let $A/Z(A)$ be a minimal ideal of $L/Z(A)$. Then $A = A^2 + Z(A)$ and A^2 is quasi-minimal in L .*

Proof. Let $P = A^2$ and $\bar{L} = L/Z(A)$. Then \bar{P} is an ideal of \bar{L} and \bar{A} is minimal, so $\bar{P} = \bar{0}$ or \bar{A} . The former implies that \bar{A} is abelian, a contradiction. Hence $\bar{P} = \bar{A}$, so $A = P + Z(A) = A^2 + Z(A)$. Also, $P = A^2 = P^2$ and $[Z(P), A] = [Z(P), P] + [Z(P), Z(A)] = 0$, so $P \cap Z(A) = Z(P)$. Thus $P/Z(P) = P/P \cap Z(A) \cong P + Z(A)/Z(A) = A/Z(A)$ is a minimal ideal of $L/Z(P)$. \square

Proposition 3.3 *Let A be quasi-minimal in L and B be an ideal of L . Then either $A \subseteq B$ or $A \subseteq C_L(B)$.*

Proof. Clearly $A \cap B + Z(A)/Z(A)$ is an ideal of $L/Z(A)$ contained in $A/Z(A)$, so $A \cap B + Z(A) = A$ or $A \cap B + Z(A) = Z(A)$. The former implies that $A = A^2 \subseteq A \cap B \subseteq A$, whence $A = A \cap B$ and $A \subseteq B$. The latter yields that $A \cap B \subseteq Z(A)$, giving $[A, B] = [A^2, B] \subseteq [A, [A, B]] \subseteq [A, A \cap B] \subseteq [A, Z(A)] = 0$ and so $A \subseteq C_L(B)$. \square

The *quasi-minimal components* of L are its quasi-minimal ideals. Write $MComp(L)$ for the set of quasi-minimal components of L , and let $E^\dagger(L)$ be the subalgebra generated by them. Then $E^\dagger(L)$ is a characteristic ideal of L , by Lemma 1.1.

Corollary 3.4 $E^\dagger(L) \subseteq C_L(R)$.

Proof. Let $A \in MComp(L)$ and put $B = R$ in Proposition 3.3. Then either $A \subseteq R$ or $A \subseteq C_L(R)$. But the former is impossible, since $A^2 = A$, whence $A \subseteq C_L(R)$. \square

Corollary 3.5 *Distinct quasi-minimal components of L commute, so*

$$E^\dagger(L) = \sum_{P \in MComp(L)} P,$$

where $[P, Q] = 0$ and $P \cap Q \subseteq Z(R)$ for all $P, Q \in MComp(L)$.

Proof. This first assertion follows directly from Proposition 3.3. But then $P \cap Q \subseteq Z(P) \cap Z(Q) \subseteq N$ and $[P, R] = [Q, R] = 0$, using Corollary 3.4. Hence $P \cap Q \subseteq Z(R)$. \square

Lemma 3.6 *If B is an ideal of L , then $MComp(B) \subseteq MComp(L) \cap B$. Moreover, if B is regular, then this is an equality.*

Proof. Let A be a quasi-minimal ideal of B . Then A is a quasi-minimal ideal of L , by Lemma 3.1. Thus $MComp(B) \subseteq MComp(L) \cap B$.

Now suppose that B is regular, and let $A \in MComp(L) \cap B$, so A is a quasi-minimal ideal of L and $A \subseteq B \cap C_L(N)$, by Corollary 3.4. Let $C/Z(A)$ be a minimal ideal of $B/Z(A)$ with $C \subseteq A$. Then $C^2 \subseteq Z(A)$ or $C^2 + Z(A) = C$. The former implies that $C^3 = 0$, and hence that C is a nilpotent ideal of B . If B is nilregular, it follows from Theorem 2.1 that $C \subseteq N$, whence $[C, A] = 0$ and $C \subseteq Z(A)$, a contradiction. Similarly, if B is solregular, then $C \subseteq R(B) \subseteq R(L)$, by Theorem 2.2. But then $[C, A] = 0$, by Corollary 3.4, since $A \in E^\dagger(L)$, leading to the same contradiction. Hence $C^2 + Z(A) = C$. But now

$$[L, C] = [L, C^2 + Z(A)] \subseteq [[L, C], C] + Z(A) \subseteq [B, C] + Z(A) \subseteq C,$$

so C is an ideal of L . But $A/Z(A)$ is a minimal ideal of $L/Z(A)$, so $C = Z(A)$ or $C = A$. It follows that $A/Z(A)$ is a minimal ideal of $B/Z(A)$ and $A^2 = A$. Thus $A \in MComp(B)$. \square

EXAMPLE 3.1 *Note that if B is not regular then the inclusion in Lemma 3.6 can be strict. For, let L be as in Example 2.4. Then \mathcal{O}_m has a unique maximal ideal \mathcal{O}_m^+ and $A^+ = sl(2) \otimes \mathcal{O}_m^+$ is the unique maximal ideal of A (and is nilpotent). Hence $MComp(A) \subseteq A^+ \neq A$, whereas $MComp(L) = A$.*

Proposition 3.7 *Let L be a Lie algebra in which $C_L(N)$ is regular. Put $Z = Z(N)$, $\overline{L} = L/Z$, $\overline{S} = Soc(\overline{C_L(N)})$. Then $E^\dagger(L) = S^2$ and $S = E^\dagger(L) + Z$.*

Proof. Let $H = C_L(N)$. Then $R(\overline{H}) = 0$, by Theorem 2.7. Hence each minimal ideal of \overline{H} is quasi-minimal in \overline{H} , and so is a quasi-minimal component of \overline{H} . Thus $\overline{S} \subseteq E^\dagger(\overline{H})$. Let $\overline{K} \in MComp(\overline{H}) \subseteq MComp(\overline{L})$, by Lemma 3.6. Hence $K/Z(K)$ is a quasi-minimal ideal of $L/Z(K)$, by Lemma 2.9. Then $K = K^2 + Z$ with K^2 quasi-minimal in L , since $Z(K) = Z$ by Lemma 3.2. Hence $K^2 \in MComp(L)$, so $S \subseteq E^\dagger(L) + Z$.

Let $P \in \text{MComp}(L)$. Then $P \subseteq H$ since $E^\dagger(L) \subseteq H$, by Corollary 3.4. Hence $P \in \text{MComp}(L) \cap H = \text{MComp}(H)$, by Lemma 3.6. Hence \overline{P} is a minimal ideal of \overline{H} , so $P \subseteq S$. Thus $S = E^\dagger(L) + Z$ and $E^\dagger(L) = S^2$. \square

We define the *quasi-minimal radical* of L to be $N^\dagger(L) = N + E^\dagger(L)$. From now on we will denote $N^\dagger(L)$ simply by N^\dagger . Then this has the property we are seeking.

Theorem 3.8 *If L is a Lie algebra, over any field F , with nilradical N , then $C_L(N^\dagger) = Z(N)$. In particular, $C_L(N^\dagger) \subseteq N^\dagger$.*

Proof. Let $C = C_L(N^\dagger)$. Then $Z(N) \subseteq C$, by Corollary 3.4. Suppose that $Z(N) \neq C$ and let $A/Z(N)$ be a minimal ideal of $L/Z(N)$ with $A \subseteq C$. Then $[A, Z(N)] \subseteq [C, N^\dagger] = 0$, so $Z(N) \subseteq Z(A)$. Thus $A = Z(A)$ or $Z(A) = Z(N)$. The former implies that $A \subseteq N$. But $[A, N] \subseteq [C, N^\dagger] = 0$, so $A \subseteq Z(N)$, a contradiction. The latter implies that $A^2 \subseteq E^\dagger \subseteq N^\dagger$, by Lemma 3.2. Hence $A^3 \subseteq [C, N^\dagger] = 0$, so $A \subseteq N$, which leads to the same contradiction as before. The result follows. \square

Proposition 3.9 *Let L be a Lie algebra in which N^\dagger is regular. Then $N^\dagger(N^\dagger) = N^\dagger$.*

Proof. Clearly $N^\dagger(N^\dagger) \subseteq N^\dagger$. But $E^\dagger(L) \subseteq E^\dagger(N^\dagger)$, by putting $B = N^\dagger$ in Lemma 3.6, and, clearly, $N \subseteq N(N^\dagger)$, giving the reverse inclusion. \square

EXAMPLE 3.2 *Again, Proposition 3.9 does not hold if N^\dagger is not regular. For, let L be as in Example 2.4. Then $N^\dagger = A$, but $N^\dagger(N^\dagger) = A^+$.*

Next we investigate the behaviour of N^\dagger with respect to factor algebras, direct sums and ideals.

Proposition 3.10 *Let L be a Lie algebra over any field, and let I be an ideal of L . Then*

$$\frac{N^\dagger(L) + I}{I} \subseteq N^\dagger\left(\frac{L}{I}\right).$$

Proof. Clearly $N(L) + I/I \subseteq N(L/I)$. Let A be a quasi-minimal ideal of L , so $A/Z(A)$ is a minimal ideal of $L/Z(A)$ and $A^2 = A$. Put $C = C_L(A + I/I)$. Then $Z(A) \subseteq C \cap A \subseteq A$, so $C \cap A = A$ or $C \cap A = Z(A)$. The former implies that $A = A^2 \subseteq I$, whence $A + I/I \subseteq N(L/I)$. If the latter holds,

then $C = C \cap (A + I) = C \cap A + I = Z(A) + I$ and $A \cap I \subseteq A \cap C = Z(A)$, whence

$$\frac{A + I/I}{Z(A + I/I)} \cong \frac{A + I}{C} = \frac{A + I}{Z(A) + I} \cong \frac{A}{Z(A) + A \cap I} = \frac{A}{Z(A)}$$

and

$$\left(\frac{A + I}{I} \right)^2 = \frac{A + I}{I}.$$

Thus $A + I/I$ is a quasi-minimal ideal of L/I and

$$\frac{E^\dagger(L) + I}{I} \subseteq E^\dagger\left(\frac{L}{I}\right).$$

The result follows. \square

The above inclusion can be strict, as we shall see later.

Proposition 3.11 *Let L be a Lie algebra over any field, and suppose that $L = I \oplus J$, where I, J are ideals of L . Then $N^\dagger(L) = N^\dagger(I) \oplus N^\dagger(J)$.*

Proof. It is easy to see that $N^\dagger(I) \oplus N^\dagger(J) \subseteq N^\dagger(L)$. Let π_I, π_J be the projection maps onto I, J respectively. Then $N(L) = \pi_I(N(L)) \oplus \pi_J(N(L))$. Clearly $\pi_I(N(L)) \subseteq N(I)$ and $\pi_J(N(L)) \subseteq N(J)$, so $N(L) \subseteq N(I) \oplus N(J)$.

Let A be a quasi-minimal ideal of L , so $A/Z(A)$ is a minimal ideal of L and $A^2 = A$. Then

$$A = A^2 \subseteq [A, I \oplus J] = [A, I] \oplus [A, J] \subseteq A,$$

so $A = [A, I] \oplus [A, J]$. Since $A = A^2 = [A, I]^2 + [A, J]^2$, we also have that $[A, I]^2 = [A, I]$ and $[A, J]^2 = [A, J]$. Now $[A, I] + Z(A) = Z(A)$ or A . The former implies that $[A, I] \subseteq Z(A)$, which gives that $[A, I] = [A, I]^2 = 0$. The latter yields that $A/Z(A) \cong [A, I]/Z(A) \cap [A, I]$. Now $Z(A) \cap [A, I] \subseteq Z([A, I])$, so $Z([A, I]) = [A, I]$ or $Z(A) \cap [A, I]$. The former gives $[A, I] = [A, I]^2 = 0$ again, whereas the latter yields that $[A, I]/Z[A, I]$ is quasi-minimal and $[A, I] \in E^\dagger(I)$.

Similarly $[A, J] = 0$ or else $[A, J] \in E^\dagger(J)$. It follows that $E^\dagger(L) \subseteq E^\dagger(I) \oplus E^\dagger(J)$, whence the result. \square

Proposition 3.12 *Let L be a Lie algebra over any field, and let I be a nilregular ideal of L . Then $N^\dagger(I) \subseteq N^\dagger(L)$.*

Proof. Since I is nilregular, we have that $N(I) \subseteq N(L)$, by Theorem 2.1 (i). Also, $E^\dagger(I) \subseteq E^\dagger(L)$, by Lemma 3.6, whence the result. \square

The following result describes the ideals of L contained in E^\dagger .

Proposition 3.13 *Let A be an ideal of L with $A \subseteq E^\dagger(L)$. Then $A = P_1 + \dots + P_k + Z(A)$, where P_i is a quasi-minimal component of L for $1 \leq i \leq k$.*

Proof. Let $E^\dagger(L) = P_1 + \dots + P_n$, where P_i is a quasi-minimal component of L for each $1 \leq i \leq n$. Then $P_i \subseteq A$ or $P_i \subseteq C_L(A)$ for each $i = 1, \dots, n$, by Proposition 3.3. Let $P_i \subseteq A$ for $1 \leq i \leq k$ and $P_i \not\subseteq A$ for $k+1 \leq i \leq n$. Then $A \cap (P_{k+1} + \dots + P_n) \subseteq Z(A)$, so $A = (P_1 + \dots + P_k) + Z(A)$. \square

Finally we give two further characterisations of N^\dagger , valid over any field. Recall that A/B is a chief factor of L if B is an ideal of L and A/B is a minimal ideal of L/B .

Theorem 3.14 *Let L be a Lie algebra, over any field F , with radical R . Then*

$$N^\dagger = \cap \{A + C_L(A/B) \mid A/B \text{ is a chief factor of } L\}.$$

Proof. Denote the given intersection by I , let A/B be a chief factor of L and let P be a quasi-minimal component of L . Then $P \subseteq A$ or $P \subseteq C_L(A)$, by Proposition 3.3. Hence $E^\dagger \subseteq I$. Moreover, $N \subseteq I$, by [2, Lemma 4.3], so $N^\dagger \subseteq I$.

If P is a quasi-minimal component of L then $P/Z(P)$ is a chief factor of L . Also, if $C = C_L(P/Z(P))$ we have $[C, P] = [C, P^2] \subseteq [[C, P], P] \subseteq [Z(P), P] = 0$, so $C = C_L(P)$ and $N \subseteq C$, by Corollary 3.4. Hence $I \subseteq P + C_L(P/Z(P)) = P + C_L(P)$. Now, if P, Q are quasi-minimal components of L , then

$$(P + C_L(P)) \cap (Q + C_L(Q)) = P + Q + C_L(P) \cap C_L(Q),$$

since $P \subseteq C_L(Q)$ and $Q \subseteq C_L(P)$. It follows that $I \subseteq N^\dagger + C_L(E^\dagger)$ and $I = N^\dagger + I \cap C_L(E^\dagger)$.

If

$$0 = N_0 \subset N_1 \subset \dots \subset N_k = N$$

is part of a chief series for L then $I \subseteq \cap_{i=1}^k C_L(N_i/N_{i-1})$, so I acts nilpotently on N . Suppose that $N \subset I \cap C_L(E^\dagger)$. Let A/N be a minimal ideal of L/N with $A \subseteq I \cap C_L(E^\dagger)$. Then $A^2 \subseteq N$ or $A^2 + N = A$. The former

implies that $A \subseteq N$, since A acts nilpotently on N , a contradiction. Hence $A = A^2 + N \subseteq A^r + N$ for all $r \geq 1$. But now

$$[A, N] \subseteq [A^r + N, N] \subseteq N(\text{ad}, A)^r + N^r,$$

so $[A, N] = 0$, whence $A \subseteq C_L(E^\dagger) \cap C_L(N) = C_L(N^\dagger) = Z(N)$, by Theorem 3.8, a contradiction again. Thus $I \cap C_L(E^\dagger) = N$ and $I = N^\dagger$. \square

We put

$$I_L(A/B) = \{x \in L \mid \text{ad}(x+B)|_{A/B} = \text{ad}(a+B)|_{A/B} \text{ for some } a \in A\}.$$

The map $\text{ad}(x+B)|_{A/B}$ is called the inner derivation *induced* by x on A/B . Then $I_L(A/B) = A + C_L(A/B)$, by [16, Lemma 1.4 (i)], so we have the following corollary.

Corollary 3.15 *Let L be a Lie algebra over any field F . Then N^\dagger is the set of all elements of L which induce an inner derivation on every chief factor of L .*

4 The generalised nilradical of L

We define the *generalised nilradical* of L , $N^*(L)$, by

$$\frac{N^*(L)}{N} = \text{Soc}_{L/N} \left(\frac{N + C_L(N)}{N} \right)$$

As usual we denote $N^*(L)$ simply by N^* . The following result shows that this is, in fact, the same as the quasi-nilpotent radical.

Theorem 4.1 *Let L be a Lie algebra with nilradical N over any field. Then $N^* = N^\dagger$.*

Proof. Put $C = C_L(N)$. Let $A/Z(A)$ be a minimal ideal of $L/Z(A)$ for which $A^2 = A$. Then $Z(A) \subseteq A \cap N$, so $A \cap N = A$ or $A \cap N = Z(A)$. the former implies that $A \subseteq N$, which is a contradiction, so the latter holds. It follows that $(A + N)/N \cong A/A \cap N = A/Z(A)$, so $(A + N)/N$ is a minimal ideal of L/N . Moreover, $[A, N] = [A^2, N] \subseteq [A, [A, N]] \subseteq [A, Z(A)] = 0$, so $A \subseteq C$ and $(A + N)/N \subseteq N^*/N$. Hence $N^\dagger \subseteq N^*$.

Now let A/N be a minimal ideal of L/N with $A \subseteq N + C$. Then $A = N + A \cap C$. Now $Z(A \cap C) = Z(N)$, by Lemma 2.9, so $A/N \cong A \cap C/N \cap C = A \cap C/Z(N) = A \cap C/Z(A \cap C)$. It follows that $A \cap C/Z(A \cap C)$

is a minimal ideal of $L/Z(A \cap C)$. Thus $(A \cap C)^2$ is a quasi-minimal ideal of L , by Lemma 3.2. Moreover, $(A \cap C)^2 + Z(N) = Z(N)$ or $A \cap C$. The former implies that $(A \cap C)^2 \subseteq Z(N)$, which yields that $(A \cap C)^3 = 0$ and $A \cap C \subseteq N$, a contradiction. Hence $A \cap C = (A \cap C)^2 + Z(N) \subseteq N^\dagger$, and so $A \subseteq N^\dagger$. This shows that $N^* \subseteq N^\dagger$. \square

This last result together with Theorem 3.8 gives the following.

Theorem 4.2 *Let L be a Lie algebra over any field F . Then $L/Z(N)$ is isomorphic to a subalgebra of $\text{Der}(N^*)$, and N^*/N is a direct sum of minimal ideals of L/N which are simple or irregular.*

Proof. The isomorphism results from the map $\theta : L \rightarrow \text{Der}(N^*)$ given by $\theta(x) = \text{ad } x|_{N^*}$. Let A/N be a minimal ideal of L/N with $A \subseteq A + C$. The $A = N + A \cap C$ and, as in the second paragraph of the proof of Theorem 4.1, $(A \cap C)^2$ is quasi-minimal in L , which implies that A/N cannot be abelian. It follows from Corollary 2.4 that A/N is simple or irregular. \square

Proposition 4.3 *Let L be a Lie algebra with nilradical N over a field F , and suppose that $C_L(N)$ is nilregular in L . Then*

$$\frac{N^*}{N} = \text{Soc} \left(\frac{N + C_L(N)}{N} \right).$$

Proof. Put $C = C_L(N)$, $D = N + C$. Let A/N be a minimal ideal of D/N . Then $A^2 + N = N$ or A . The former implies that $A^2 \subseteq N$, whence $A^3 \subseteq [N, N + C] \subseteq N^2$, and an easy induction shows that $A^{n+1} \subseteq N^n = 0$ for some $n \in \mathbb{N}$. It follows that A is a nilpotent ideal of D , which is an ideal of L , and thus that $A \subseteq N(D) = N + N(C) \subseteq N$, by Theorem 2.1, a contradiction. Hence $A = A^2 + N$ and

$$[L, A] = [L, A^2 + N] \subseteq [[L, A], A] + [L, N] \subseteq [D, A] + N \subseteq A,$$

so A/N is a minimal ideal of L/N inside D/N .

Now suppose that B/N is a minimal ideal of L/N inside D/N , and let A/N be a minimal ideal of D/N inside B/N . Then, by the argument in the paragraph above, A/N is an ideal of L/N , and so $A = B$. The result follows. \square

Proposition 4.4 (i) *If $C_L(N)$ is regular and $\phi(L) \cap Z(N) = 0$ then $N^*(L) = N(L) \oplus S$, where S is the socle of a maximal semisimple ideal of L .*

- (ii) Over a field of characteristic zero, $N^*(L) = N(L) \oplus S = N(L) + C_L(N)$, where S is the biggest semisimple ideal of L .

Proof. This follows from Corollary 2.8. \square

Proposition 4.5 *Let L be a Lie algebra over a field of characteristic zero and let $I \subseteq N^*(L)$ be an ideal of L . Then*

$$\frac{N^*(L)}{I} \subseteq N^*\left(\frac{L}{I}\right).$$

Proof. This is a special case of Proposition 3.10. \square

As a result of Example 3.2 we define, for each non-negative integer n , N_n^* , inductively by

$$N_0^*(L) = L \text{ and } N_n^* = N^*(N_{n-1}^*(L)) \text{ for } n > 0.$$

Clearly the series

$$L = N_0^*(L) \supseteq N_1^*(L) \supseteq \dots$$

will terminate in an equality, so we put $N_\infty^*(L)$ equal to the minimal subalgebra in this series. It is easy to see that $N_\infty^*(N_\infty^*(L)) = N_\infty^*(L)$. Then we have

Proposition 4.6 *Let $n \in \mathbb{N} \cup \{0\}$, and let I, J be ideals of the Lie algebra L over the field F . Then*

- (i) *if $N_k^*(I)$ is a nilregular ideal of $N_k^*(L)$ then $N_{k+1}^*(I)$ is a characteristic ideal of $N_k^*(L)$ for $k \geq 0$;*
- (ii) *if $I \subseteq N_n^*(L)$ is an ideal of L then $N_{n+1}^*(L)/I \subseteq N_{n+1}^*(L/I)$.*
- (iii) *if $L = I \oplus J$, then $N_k^*(L) = N_k^*(I) \oplus N_k^*(J)$ for all $k \geq 0$.*

Proof.

- (i) This follows from Theorem 2.1 (i) and Lemma 3.1.
- (ii) The case $n = 1$ is given by Proposition 4.5. So suppose that the case $n = k$ holds, where $k \geq 1$, and let $I \subseteq N_k^*(L)$. Then $I \subseteq N_{k-1}^*(L)$. Hence

$$\begin{aligned} \frac{N_{k+1}^*(L)}{I} &= \frac{N^*(N_k^*(L))}{I} \subseteq N^*\left(\frac{N_k^*(L)}{I}\right) \\ &\subseteq N^*\left(N_k^*\left(\frac{L}{I}\right)\right) = N_{k+1}^*\left(\frac{L}{I}\right). \end{aligned}$$

The result now follows by induction

- (iii) This is a straightforward induction proof: the case $k = 1$ is given by Proposition 3.11

□

Corollary 4.7 *Let $n \in \mathbb{N}$, and let I, J be ideals of the Lie algebra L over the field F . Then*

- (i) *if $N_\infty^*(I)$ is nilregular, it is a characteristic ideal of $N_\infty^*(L)$;*
- (ii) *if $I \subseteq N_\infty^*(L)$ is an ideal of L then $N_\infty^*(L)/I \subseteq N_\infty^*(L/I)$.*
- (iii) *if $L = I \oplus J$, then $N_\infty^*(L) = N_\infty^*(I) \oplus N_\infty^*(J)$.*

5 The quasi-nilpotent radical

Here we construct a radical by adjoining the quasi-simple ideals of L to the nilradical N . Since quasi-simple ideals are quasi-minimal they are characteristic in L .

Lemma 5.1 *Let $L/Z(L)$ be simple. Then $L = L^2 + Z(L)$ and L^2 is quasi-simple.*

Proof. Let $P = L^2$ and $\overline{L} = L/Z(L)$. Then \overline{P} is an ideal of \overline{L} and \overline{L} is simple, so $\overline{P} = 0$ or \overline{L} . The former implies that \overline{L} is abelian, a contradiction. Hence $\overline{P} = \overline{L}$, and so $L = P + Z(L) = L^2 + Z(L)$. Also, $P = L^2 = P^2$ and $P/Z(P) = P/P \cap Z(L) \cong (P + Z(L))/Z(L) = L/Z(L)$ is simple. □

Lemma 5.2 *Let A be a quasi-simple ideal of L and B an ideal of L . Then either $A \subseteq B$ or $A \subseteq C_L(B)$.*

Proof. Since quasi-simple ideals are quasi-minimal the result follows from Proposition 3.3. □

The *quasi-simple components* of L are its quasi-simple ideals. We will write $\text{SComp}(L)$ for the set of quasi-simple components of L , and put $\hat{E}(L) = \langle \text{SComp}(L) \rangle$, the subalgebra generated by the quasi-simple components of L . Clearly $\text{SComp}(L) \subseteq \text{MComp}(L)$, $\hat{E}(L) \subseteq E^\dagger(L)$ and $\hat{E}(L)$ is characteristic in L .

Lemma 5.3 *If B is an ideal of L , then $\text{SComp}(B) = \text{SComp}(L) \cap B$.*

Proof. If A is a quasi-simple ideal of B , it is an ideal of L since it is characteristic in B , and so $\text{SComp}(B) \subseteq \text{SComp}(L) \cap B$. The reverse inclusion is clear. \square

Proposition 5.4 *Let $P \in \text{SComp}(L)$ and let B be an ideal of L . Then $P \in \text{SComp}(B)$ or $[P, B] = 0$.*

Proof. Suppose that $[P, B] \neq 0$. We have that P is a quasi-simple ideal of L , so $P \subseteq B$, by Lemma 5.2. Hence $P \in \text{SComp}(B)$, by Lemma 5.3. \square

Corollary 5.5 *Distinct quasi-simple components of L commute, so*

$$\hat{E}(L) = \sum_{P \in \text{SComp}(L)} P,$$

where $[P, Q] = 0$ and $P \cap Q \subseteq Z(R)$ for all $P, Q \in \text{SComp}(L)$.

Proof. This follows easily as in Corollary 3.5. \square

Theorem 5.6 *Suppose that L is a Lie algebra in which $E^\dagger(L)$ is regular, then $\hat{E}(L) = E^\dagger(L)$.*

Proof. Let P be a quasi-simple ideal of L . Then $N(P)$ and $R(P)$ are ideals of $E^\dagger(L)$, by Corollary 5.5. It follows that P is a regular ideal of L and the result follows from Corollary 2.6. \square

Clearly, if L is as in Example 2.4 we have $\hat{E}(L) = 0 \neq A = E^\dagger(L)$, so Theorem 5.6 does not hold for all Lie algebras.

Corollary 5.7 *Let L be a Lie algebra in which $E^\dagger(L)$ and $C_L(N)$ are regular. Put $Z = Z(N)$, $\overline{L} = L/Z$, $\overline{S} = \text{Soc}(\overline{C_L(N)})$. Then $\hat{E}(L) = S^2$ and $S = \hat{E}(L) + Z$.*

Proof. This follows from Proposition 3.7 and Theorem 5.6. \square

We define the *quasi-nilpotent radical* of L to be $\hat{N}(L) = N + \hat{E}(L)$. From now on we will denote $\hat{N}(L)$ simply by \hat{N} . The following is an immediate consequence of Theorems 3.8 and 5.6.

Corollary 5.8 *Suppose that L is a Lie algebra in which $N^\dagger(L)$ is regular. Then $C_L(\hat{N}) = Z(N)$. In particular $C_L(\hat{N}) \subseteq \hat{N}$.*

Once more, Example 2.4 shows that the above result does not hold without some restrictions. For, if L is as in that example, then $\hat{N}(L) = 0$ and $C_L(\hat{N}(L)) = L$.

Proposition 5.9 *Let L be a Lie algebra over a field F , and let B be a nilregular ideal of L . Then $\hat{N}(B) \subseteq \hat{N}$.*

Proof. Under the given hypotheses $N(B)$ is a characteristic ideal of B (see [7]), so $N(B) \subseteq N$. Moreover, $\hat{E}(B) \subseteq \hat{E}(L)$ by Lemma 5.3. \square

Proposition 5.10 *Let L be a Lie algebra over any field. Then $\hat{N}(\hat{N}) = \hat{N}$.*

Proof. Clearly $\hat{N}(\hat{N}) \subseteq \hat{N}$. But $\hat{E}(\hat{N}) = \hat{E}(L)$, by Lemma 5.3, and, clearly, $N \subseteq N(\hat{N})$, giving the reverse inclusion. \square

Proposition 5.11 *Let L be a Lie algebra over any field, and let I be an ideal of L . Then*

$$\frac{\hat{N}(L) + I}{I} \subseteq \hat{N}\left(\frac{L}{I}\right).$$

Proof. This follows exactly as in Proposition 3.10. \square

6 Another generalisation of the nilradical

We put $\tilde{N}(L)/\phi(L) = \text{Soc}(L/\phi(L))$. We write $\tilde{N}(L)$ simply as \tilde{N} . Then we see that this radical also has our desired property.

Theorem 6.1 *Let L be a Lie algebra over any field, with nilradical N . Then $C_L(\tilde{N}) \subseteq Z(N) \subseteq \tilde{N}$.*

Proof. Put $C = C_L(\tilde{N})$. Suppose first that $\phi(L) = 0$. Then $L = N \dot{+} U$ where $N = \text{Asoc} L$ and U is a subalgebra of L , by [14, Theorems 7.3 and 7.4]. Then $C = N \dot{+} C \cap U$ and $C \cap U$ is an ideal of L . Suppose that $C \cap U \neq 0$ and let A be a minimal ideal of L with $A \subseteq C \cap U$. Then $A \subseteq \tilde{N}$, so $A^2 \subseteq [\tilde{N}, C] = 0$. Hence $A \subseteq N \cap U = 0$, a contradiction. It follows that $C = N$.

If $\phi(L) \neq 0$ we have

$$\frac{C + \phi(L)}{\phi(L)} \subseteq C_{L/\phi(L)}\left(\frac{\tilde{N}}{\phi(L)}\right) \subseteq \frac{N}{\phi(L)}.$$

Hence $C \subseteq N$, which yields $C \subseteq Z(N)$. \square

Theorem 6.2 *Let L be a ϕ -free Lie algebra over any field F and suppose that $\tilde{N}(L)$ is nilregular. Then $L/C_L(\tilde{N}(L))$ is isomorphic to a subalgebra of*

$$\mathcal{M}_r^- \oplus \left(\bigoplus_{i=1}^s \text{Der}(A_i) \right)$$

where \mathcal{M}_r is the set of $r \times r$ matrices over F , r is the dimension of the nilradical, and A_1, \dots, A_s are the simple minimal ideals of L .

Proof. Since L is ϕ -free we have that $\tilde{N}(L) = N(L) \oplus (\bigoplus_{i=1}^r A_i)$ where A_1, \dots, A_r are the non-abelian minimal ideals of L . Also, each A_i is nilregular and hence simple, by Corollary 2.4. The map $\theta : L \rightarrow \text{Der}(\tilde{N}(L))$ given by $\theta(x) = \text{ad } x|_{\tilde{N}(L)}$ is a homomorphism with kernel $C_L(\tilde{N}(L))$. But $N(L)$ is characteristic, since it is nilregular, and the A_i 's are characteristic, since they are perfect, so

$$\text{Der}(\tilde{N}(L)) = \text{Der}(N(L)) \oplus \left(\bigoplus_{i=1}^s \text{Der}(A_i) \right),$$

whence the result. \square

Proposition 6.3 $N^* \subseteq \tilde{N}$.

Proof. There is a subalgebra $U/\phi(L)$ of $L/\phi(L)$ such that $L/\phi(L) = N/\phi(L) \dot{+} U/\phi(L)$, by [14, Theorems 7.3 and 7.4]. Let A/N be a minimal ideal of L/N with $A \subseteq N + C_L(N)$. Then $A = N \dot{+} A \cap U$, so $[N, A] = [N, N + A \cap C] \subseteq \phi(L)$ and $A \cap U/\phi(L)$ is a minimal ideal of $L/\phi(L)$. Moreover, $N/\phi(L) \subseteq \text{Soc}(L/\phi(L))$, by [14, Theorem 7.4]. Hence $A/N \subseteq \text{Soc}(L/\phi(L))$, and so $N^* \subseteq \tilde{N}$. \square

In general we can have $N^* \subset \tilde{N}$ and $\tilde{N}(\tilde{N}) \subset \tilde{N}$, as we will show below. Recall that the category \mathcal{O} is a mathematical object in the representation theory of semisimple Lie algebras. It is a category whose objects are certain representations of a semisimple Lie algebra and morphisms are homomorphisms of representations. The formal definition and its properties can be found in [5]. As in other artinian module categories, it follows from the existence of enough projectives that each $M \in \mathcal{O}$ has a *projective cover* $\pi : P \rightarrow M$. Here π is an epimorphism and is essential, meaning that no proper submodule of the projective module P is mapped onto M . Up to isomorphism the module P is the unique projective having this property (see [5, page 62]).

EXAMPLE 6.1 So let S be a finite-dimensional simple Lie algebra over a field F of prime characteristic, let P be the projective cover for the trivial irreducible S -module and let R be the radical of P . Then R is a faithful irreducible S -module and P/R is the trivial irreducible S -module. Let $T = P \rtimes S$ be the semidirect sum of P and S . Then $T^2 = R \rtimes S$ is a primitive Lie algebra of type 1 and $\dim(T/T^2) = 1$, say $T = T^2 + Fx$. Put $L = T + Fy$ where $[x, y] = y$ and $[T^2, y] = 0$.

Then $\phi(T) \subseteq T^2$, so $\phi(T)$ is an ideal of L and $\phi(T) \subseteq \phi(L)$, by [14, Lemma 4.1]. But $\phi(L) \subseteq T$ and, if M is a maximal subalgebra of T then $M + Fy$ is a maximal subalgebra of L , so $\phi(L) = \phi(T) = R$. Also $\text{Soc}(L/R) = (T^2 + Fy)/R$, so $\tilde{N}(L) = T^2 \oplus Fy$. However, $N(L) = R \oplus Fy$ and $C_L(N(L)) = N(L)$, so $N^*(L) = N(L) \neq \tilde{N}(L)$.

Moreover, $\phi(\tilde{N}(L)) = 0$, so $\tilde{N}(\tilde{N}(L)) = \text{Soc}(\tilde{N}(L)) = R \oplus Fy \neq \tilde{N}(L)$.

Notice that we also have $N^*(L)/\phi(L) = N(L)/R \cong Fy$, whereas $N^*(L/\phi(L)) = T^2 + Fy/R$. Hence the inclusions in Propositions 3.10, 4.5, 4.6 and Corollary 4.7 can be strict.

Note that a similar example can be constructed in characteristic p . Let L be a finite-dimensional restricted Lie algebra over a field F of prime characteristic, and let $u(L)$ denote the restricted universal enveloping algebra of L . Then every restricted L -module is a $u(L)$ -module and vice versa, and so there is a bijection between the irreducible restricted L -modules and the irreducible $u(L)$ -modules. In particular, as $u(L)$ is finite-dimensional, every irreducible restricted L -module is finite-dimensional. So, in the above example we could take S to be a restricted simple Lie algebra, as the projective cover of the trivial S -module again exists.

Proposition 6.4 *If I is an ideal of L then*

$$\frac{\tilde{N} + I}{I} \subseteq \tilde{N} \left(\frac{L}{I} \right).$$

Moreover, if $I \subseteq \phi(L)$, then $\tilde{N}(L)/I = \tilde{N}(L/I)$.

Proof. Let $A/\phi(L)$ be a minimal ideal of $L/\phi(L)$. Then

$$\frac{A + I/I}{\phi(L) + I/I} \cong \frac{A + I}{\phi(L) + I} \cong \frac{A}{A \cap (\phi(L) + I)}.$$

Now $\phi(L) \subseteq A \cap (\phi(L) + I)$, so $A \cap (\phi(L) + I) = A$ or $\phi(L)$. But $\phi(L) + I/I \subseteq \phi(L/I)$, so the former implies that $A + I/I = \phi(L/I)$ and $A + I/I \subseteq \tilde{N}(L/I)$.

If the latter holds then $A \cap I \subseteq \phi(L)$. But now, $\phi(L)/A \cap I = \phi(L/A \cap I)$, by [14, Proposition 4.3], so

$$\frac{A/A \cap I}{\phi(L/A \cap I)} = \frac{A/A \cap I}{\phi(L)/A \cap I} \cong \frac{A}{\phi(L)}.$$

It follows that $A/A \cap I \subseteq \tilde{N}(L/A \cap I)$, whence $A + I/I \subseteq \tilde{N}(L/I)$.

The second assertion follows from the definition of \tilde{N} and the fact that $\phi(L/I) = \phi(L)/I$. \square

Proposition 6.5 $\tilde{N}(L)/\phi(L) = N^*(L/\phi(L))$.

Proof. Suppose first that $\phi(L) = 0$. Then $\tilde{N}(L)$ is the socle of L . Now $N(L) = \text{Asoc}(L)$, by [14, Theorem 7.4]. Also, if A is a minimal ideal of L with $A \not\subseteq N(L) = N$, then $[A, N] \subseteq A \cap N = 0$, so $A \subseteq C_L(N)$. Hence $\tilde{N}(L) \subseteq N^*(L)$.

If $\phi(L) \neq 0$ the above shows that $\tilde{N}(L/\phi(L)) \subseteq N^*(L/\phi(L))$. The result now follows from Propositions 6.3 and 6.4. \square

Proposition 6.6 *If $L = I \oplus J$, then $\tilde{N}(L) = \tilde{N}(I) \oplus \tilde{N}(J)$.*

Proof. We have that $N(L) = N(I) \oplus N(J)$ and $\phi(L) = \phi(I) \oplus \phi(J)$ by [14, Theorem 4.8]. Let $A/\phi(L)$ be a minimal ideal of $L/\phi(L)$ and suppose that $A \not\subseteq N(L)$. Then $A = A^2 + \phi(L)$. But $\phi(L) = \phi(I) \oplus \phi(J)$, by [14, Theorem 4.8], so

$$A = A^2 + \phi(I) + \phi(J) = [A, I] + \phi(I) + [A, J] + \phi(J).$$

Hence

$$\frac{A}{\phi(L)} \cong \frac{[A, I] + \phi(I)}{\phi(I)} \oplus \frac{[A, J] + \phi(J)}{\phi(J)}.$$

It is easy to see that the direct summands are minimal ideals of $I/\phi(I)$ and $J/\phi(J)$ respectively, so $\tilde{N}(L) \subseteq \tilde{N}(I) \oplus \tilde{N}(J)$. Also, if $A/\phi(I)$ is a minimal ideal of $I/\phi(I)$, then $A + \phi(J)/\phi(L)$ is a minimal ideal of $L/\phi(L)$, so $\tilde{N}(I) \subseteq \tilde{N}(L)$. Similarly $\tilde{N}(J) \subseteq \tilde{N}(L)$, which gives the result. \square

As a result of Example 6.1 we define, for each non-negative integer n , $\tilde{N}_n(L)$ inductively by

$$\tilde{N}_0(L) = L \text{ and } \tilde{N}_n(L) = \tilde{N}(\tilde{N}_{n-1}(L)) \text{ for } n > 0.$$

Clearly the series

$$L = \tilde{N}_0(L) \supseteq \tilde{N}_1(L) \supseteq \dots$$

will terminate in an equality, so we put $\tilde{N}_\infty(L)$ equal to the minimal subalgebra in this series. It is easy to see that $\tilde{N}_\infty(\tilde{N}_\infty(L)) = \tilde{N}_\infty(L)$.

Proposition 6.7 *Let $n \in \mathbb{N} \cup \{0\}$, and let I, J be ideals of the Lie algebra L over a field F .*

- (i) *If $I \subseteq \phi(\tilde{N}_{n-1}(L))$ then $\tilde{N}_n(L/I) = \tilde{N}_n(L)/I$.*
- (ii) *$N(\tilde{N}_n(L)) \subseteq N(\tilde{N}_{n+1}(L))$ for each $n \geq 0$.*
- (iii) *If $\tilde{N}_\infty(L)$ is nilregular, then $\phi(\tilde{N}_{n+1}(L)) \subseteq \phi(\tilde{N}_n(L))$ for each $n \geq 0$.*
- (iv) *If $\tilde{N}_\infty(L)$ is nilregular then $N(\tilde{N}_n(L)) = N(L)$ and $\tilde{N}_n(L)$ is an ideal of L for all $n \geq 0$.*
- (v) *If $N^*(L)$ is nilregular then $N^*(L) \subseteq \tilde{N}_n(L)$ for each $n \geq 0$.*
- (vi) *If $\tilde{N}_n(L)$ is nilregular and $\phi(\tilde{N}_n(L)) = 0$ then $\tilde{N}_{n+1}(L) = N^*(L)$.*
- (vii) *If $N^*(L)$ is nilregular then $C_L(\tilde{N}_n(L)) = Z(N(L))$.*
- (viii) *If F has characteristic zero, then $\tilde{N}_n(I) \subseteq \tilde{N}_n(L)$.*
- (ix) *If F has characteristic zero, then $\tilde{N}_n(L) + I/I \subseteq \tilde{N}_n(L/I)$.*
- (x) *If $L = I \oplus J$ then $\tilde{N}_n(L) = \tilde{N}_n(I) \oplus \tilde{N}_n(J)$.*

Proof.

- (i) The case $n = 1$ is given by Proposition 6.4. A straightforward induction argument then yields the general case.
- (ii) We have that $N(L) \subseteq \tilde{N}(L)$, by [14, Theorem 7.4], whence $N(L) \subseteq N(\tilde{N}(L))$. Thus $N(\tilde{N}(L)) \subseteq N(\tilde{N}_2(L))$, and a simple induction argument gives the general result.
- (iii) Put $\tilde{N}_i = \tilde{N}_i(L)$. Then

$$\frac{\tilde{N}_{n+1}}{\phi(\tilde{N}_n)} = \bigoplus_{i=1}^r \frac{A_i}{\phi(\tilde{N}_n)},$$

where each direct summand is a minimal ideal of $\tilde{N}_n/\phi(\tilde{N}_n)$. Now

$$N\left(\frac{A_i}{\phi(\tilde{N}_n)}\right) \subseteq N\left(\frac{\tilde{N}_n}{\phi(\tilde{N}_n)}\right) = \frac{N(\tilde{N}_n)}{\phi(\tilde{N}_n)}$$

and $N(\tilde{N}_n) \subseteq N(\tilde{N}_\infty)$ by (ii), so the direct summands are nilregular, and hence are abelian or simple, by Corollary 2.4. It follows that they are ϕ -free, and thus, so is $\tilde{N}_{n+1}/\phi(\tilde{N}_n)$. The result follows.

- (iv) Consider the first assertion: it clearly holds for $n = 0$. Suppose that $\tilde{N}_\infty(L)$ is nilregular and that the result holds for $k \leq n$ ($n \geq 0$). Then $\tilde{N}_k(L)$ is nilregular for all $k \geq 0$, by (ii). It follows from [9, Corollary 1] that $N(\tilde{N}_n(L))$ is a characteristic ideal of $\tilde{N}_n(L)$, and hence an ideal of $\tilde{N}_{n-1}(L)$. Thus $N(\tilde{N}_n(L)) = N(\tilde{N}_{n-1}(L))$, and so $N(\tilde{N}_n(L)) = N(L)$ by the inductive hypothesis, which proves the first assertion.

Put $\tilde{N}_n = \tilde{N}_n(L)$, $\phi_n = \phi(\tilde{N}_n)$ and let A/ϕ_n be a minimal ideal of \tilde{N}_n/ϕ_n . If $A \not\subseteq N(\tilde{N}_n)$, then A/ϕ_n is a perfect subideal of L/ϕ_n and so an ideal of L/ϕ_n , by Lemma 1.1. The result follows.

- (v) The case $n = 1$ is Proposition 6.3. So suppose that $N^*(L) \subseteq \tilde{N}_k(L)$ for some $k \geq 1$. Then

$$N^*(L) = N^*(N^*(L)) \subseteq N^*(\tilde{N}_k(L)) \subseteq \tilde{N}_{k+1}(L),$$

by Propositions 3.9, 3.12 and 6.3.

- (vi) If $\phi(\tilde{N}_n(L)) = 0$ then

$$\tilde{N}_{n+1}(L) \subseteq N^*(\tilde{N}_n(L)) \subseteq N^*(L) \subseteq \tilde{N}_{n+1}(L),$$

since $\tilde{N}_n(L)$ is nilregular (and hence so is $N^*(L)$), by Propositions 6.5, 3.12 and (v) above.

- (vii) Using (v) above we have that $C_L(\tilde{N}_n(L)) \subseteq C_L(N^*(L)) = Z(N)$, by Theorem 3.8.

- (viii) We have $\phi(I) \subseteq \phi(L)$, by [14, Corollary 3.3], so $\tilde{N}(L/\phi(I)) = \tilde{N}(L)/\phi(I)$. Now

$$\tilde{N}(I)/\phi(I) = N^*(I/\phi(I)) \subseteq N^*(L/\phi(I)) \subseteq \tilde{N}(L/\phi(I)) = \tilde{N}(L)/\phi(I),$$

by Propositions 6.5, 3.12 and 6.3. Hence $\tilde{N}(I) \subseteq \tilde{N}(L)$. Then a simple induction proof shows that $\tilde{N}_n(I) \subseteq \tilde{N}_n(L)$.

- (ix) The case $n = 1$ is given by Proposition 6.4. Suppose it holds for some $k \geq 1$. Then

$$\begin{aligned} \frac{\tilde{N}_{k+1}(L) + I}{I} &= \frac{\tilde{N}(\tilde{N}_k(L)) + I}{I} \subseteq \frac{\tilde{N}(\tilde{N}_k(L) + I) + I}{I} \\ &\subseteq \tilde{N}\left(\frac{\tilde{N}_k(L) + I}{I}\right) \subseteq \tilde{N}\left(\tilde{N}_k\left(\frac{L}{I}\right)\right) = \tilde{N}_{k+1}\left(\frac{L}{I}\right), \end{aligned}$$

by (viii) and Proposition 6.4.

- (x) The case $n = 1$ is given by Proposition 6.6. A straightforward induction argument then gives the general result.

□

Corollary 6.8 *Let I, J be ideals of L .*

- (i) *If $I \subseteq \phi(\tilde{N}_\infty(L))$ then $\tilde{N}_\infty(L/I) = \tilde{N}_\infty(L)/I$.*
- (ii) *If $\tilde{N}_\infty(L)$ is nilregular the $N(\tilde{N}_\infty(L)) = N(L)$ and $\tilde{N}_\infty(L)$ is an ideal of L .*
- (iii) *If $N^*(L)$ is nilregular then $N^*(L) \subseteq \tilde{N}_\infty(L)$.*
- (iv) *If $\tilde{N}_\infty(L)$ is nilregular and $\phi(\tilde{N}_\infty(L)) = 0$ then $\tilde{N}_\infty(L) = N^*(L)$.*
- (v) *If $N^*(L)$ is nilregular then $C_L(\tilde{N}_\infty(L)) = Z(N(L))$.*
- (vi) *If F has characteristic zero, then $\tilde{N}_\infty(I) \subseteq \tilde{N}_\infty(L)$.*
- (vii) *If F has characteristic zero, then $\tilde{N}_\infty(L) + I/I \subseteq \tilde{N}_\infty(L/I)$;*
- (viii) *If $L = I \oplus J$ then $\tilde{N}_\infty(L) = \tilde{N}_\infty(I) \oplus \tilde{N}_\infty(J)$.*

If S is a subalgebra of L the *core* of S , S_L , is the biggest ideal of L contained in S . The following is an analogue of a result for groups given by Vasil'ev et al. in [17].

Theorem 6.9 *Let L be a Lie algebra over any field. Then the core of the intersection of all maximal subalgebras such that $L = M + \tilde{N}(L)$ is equal to $\phi(L)$.*

Proof. Put P equal to the intersection of all maximal subalgebras such that $L = M + \tilde{N}$. Clearly $\tilde{N} \not\subseteq \phi(L)$ and $\phi(L) \subseteq P_L$. Factor out $\phi(L)$ and suppose that $P_L \neq 0$. Let A be a minimal ideal of L contained in P_L . Then $A \subseteq \tilde{N}(L)$.

Since $\phi(L) = 0$ there is a maximal subalgebra of L such that $A \not\subseteq M$. If $L = \tilde{N}(L) + M$ we have $A \subseteq P_L \subseteq M$, a contradiction. If not, then $A \subseteq \tilde{N}(L) \subseteq M$, a contradiction again. Hence $P_L = 0$.

It follows that $P_L \subseteq \phi(L)$, whence the result.

□

References

- [1] M. ASCHBACHER, ‘Finite Group Theory’, Cambridge University Press (2000), ISBN 978-0-521-78675-1.
- [2] D.W. BARNES AND H.M. GASTNEAU-HILLS, ‘On the theory of soluble Lie algebras’, *Math. Z.* **106** (1968), 1086–1090.
- [3] R.E. BLOCK, ‘Differentiably simple algebras’, *Bull. Amer. Math. Soc.* **74** (1968), 433 – 459.
- [4] L. A. BOKUT, ‘On nilpotent Lie algebras’ *Algebra and Logic* **10** (1971), 135–168.
- [5] J.E. HUMPHREYS, ‘Representations of semisimple Lie algebras in the BGG Category \mathcal{O} ’, AMS Graduate Studies in Mathematics, **94** (2008).
- [6] N.JACOBSON, ‘Lie algebras’, Dover, New York, (1979).
- [7] D. V. MAKSIMENKO, ‘On action of outer derivations on nilpotent ideals of Lie algebras’, *Algebra Discrete Math.* **1** (2009), 74–82.
- [8] V.I. MURASHKA AND A.F. VASILEV, ‘Generalised Fitting subgroups of finite groups’, arXiv:1310.7445.
- [9] A.P. PETRAVCHUK, ‘On behavior of solvable ideals of Lie algebras under outer derivations’, *Comm. Alg.* **38** (2010), 2311–2316.
- [10] E. SCHENKMAN, ‘A theory of subinvariant Lie algebras’, *Amer. J. Math.* **73** (1951), 453–474.
- [11] G.B. SELIGMAN, ‘Characteristic ideals and the structure of Lie algebras’, *Proc. Amer. Math. Soc.* **8**, No. **1** (1957), 159–164.
- [12] E.L. STITZINGER, ‘Frattini subalgebras of a class of solvable Lie algebras’, *Pacific J. Math.* **34**, No. 1 (1970), 177–182.
- [13] H. STRADE, ‘Simple Lie algebras over fields of positive characteristic, Volume 1: Structure Theory’, De Gruyter Expositions in Mathematics **38** (2004) ISBN (Online): 9783110197945.
- [14] D.A. TOWERS, ‘A Frattini theory for algebras’, *Proc. London Math. Soc.* (3) **27** (1973), 440–462.

- [15] D.A. TOWERS, ‘On n -maximal subalgebras of Lie algebras’,
<http://arxiv.org/abs/1502.02865>.
- [16] D.A. TOWERS AND Z. CILOGLU, ‘On complemented non-abelian chief factors of a Lie algebra’, preprint.
- [17] A.F. VASILEV, T.I. VASILEVA AND A.V. SYROKVASHIN, ‘A note on intersection of some maximal sub- groups of finite groups’. *Probl. Mat. Fiz. Tech.* **11 (2)** (2012), 62-64. (In Russian)
- [18] P. ZUSMANOVICH, ‘Deformations of $W_1(n) \otimes A$ and modular semisimple Lie algebras with a solvable maximal subalgebra’, *J. Algebra* **268** (2003), 603–635.